An Improved Approximation Algorithm for the Asymmetric TSP with Strengthened Triangle Inequality

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Abstract. We consider the asymmetric traveling salesperson problem with γ-parameterized triangle inequality for $\gamma \in [\frac{1}{2}, 1)$. That means, the edge weights fulfill $w(u,v) \le \gamma \cdot (w(u,x) + w(x,v))$ for all nodes u, v, x. Chandran and Ram [6] recently gave the first constant factor approximation algorithm with polynomial running time for this problem. They achieve performance ratio $\frac{\gamma}{1-\gamma}$. We devise an approximation algorithm with performance ratio $\frac{1}{1-\frac{1}{2}(\gamma+\gamma^3)}$, which is better than the one by Chandran and Ram for $\gamma \in [0.6507, 1)$, that is, for the particularly interesting large values of γ.

1 Introduction

The traveling salesperson problem is a well-known NP optimization problem. Given a complete loopless graph G and a weight function w that assigns to each edge a nonnegative weight, our goal is to find a tour of minimum weight that visits each node exactly once. In general, the graph G may be directed. In this case, one also speaks of the asymmetric traveling salesperson problem (ATSP). An important and well-studied special case is the case where w is symmetric (TSP), that is, $w(u,v) = w(v,u)$ for all $u, v \in V$. In other words, the underlying graph can be considered undirected.

TSP and henceforth ATSP are both NPO-complete. Thus there is no good approximation algorithm for these two problems, unless $\mathsf{NP} = \mathsf{P}$. A natural restriction is that the weight function w should fulfill the triangle inequality

$$w(u,v) \le w(u,x) + w(x,v) \quad \text{for all } u, x, v \in V. \tag{1}$$

We call the corresponding problems Δ-ATSP and Δ-TSP in the asymmetric and symmetric case, respectively. For Δ-TSP, Christofides [7] devised a $\frac{3}{2}$ approximation algorithm with polynomial running time, whereas the best approximation algorithm for Δ-ATSP has only performance ratio $\log n$. This was shown by Frieze, Galbiati, and Maffioli [8]. See also [4] for some slight improvement. Many researchers conjecture that there is also a constant factor approximation algorithm for Δ-ATSP, but this question is still open after more than two decades.

We here consider a strengthening of the triangle inequality (1), which allows a constant factor approximation: Let γ be some constant with $\frac{1}{2} \leq \gamma < 1$. An instance of the problem $\Delta(\gamma)$-ATSP is a complete loopless directed graph G with node set V and a weight function w assigning to each edge of G a nonnegative weight. The weight function fulfills the *γ-parameterized triangle inequality*, i.e.,

$$w(u,v) \leq \gamma \cdot (w(u,x) + w(x,v)) \quad \text{for all } u,x,v \in V. \tag{2}$$

The goal is to compute a TSP tour of minimum weight.

One can also view the γ-parameterized triangle inequality as a data dependent bound. Given an instance of Δ-ATSP, we compute $\tilde{\gamma} = \max\{\frac{w(u,v)}{w(u,x)+w(x,v)}\}$ and use our algorithm to obtain better performance guarantees on instances where $\tilde{\gamma}$ is small enough. If $\tilde{\gamma} + \tilde{\gamma}^3 \leq 2 - 2/\log n$, then this is better than the $\log n$ upper bound.

1.1 Previous and new results

As mentioned above, for Δ-ATSP and Δ-TSP, there are approximation algorithms with polynomial running time achieving performance ratios $\log n$ and $\frac{3}{2}$, respectively.

Böckenhauer et al. [5] studied the symmetric traveling salesperson problem with γ-parameterized triangle inequality for $\gamma \in [\frac{1}{2}, 1)$. They achieve approximation performance $\min\{1 + \frac{2\gamma-1}{3\gamma^2-2\gamma+1}, \frac{2}{3} + \frac{\gamma}{3(1-\gamma)}\}$. Andreae and Bandelt [2] as well as Bender and Chekuri [3] considered the symmetric case with γ-parameterized triangle inequality for $\gamma \geq 1$. Combining their algorithms, we get an approximation algorithm with performance guarantee $\min\{\frac{3\gamma^2+\gamma}{2}, 4\gamma\}$.

Recently, Chandran and Ram [6] studied the asymmetric traveling salesperson problem with γ-parameterized triangle inequality for $\gamma \in [\frac{1}{2}, 1)$. They designed a constant factor approximation algorithm with performance ratio (asymptotically) $\frac{\gamma}{1-\gamma}$, in contrast to the $\log n$ upper bound for Δ-ATSP. Our Lemma 2 shows that the algorithm of Frieze, Galbiati, and Maffioli without any modifications already yields a $\frac{1}{1-\gamma}$ approximation for $\Delta(\gamma)$-ATSP. Since we even do not know whether for $\gamma = 1$ an approximation algorithm with constant performance ratio exists, studying the case $\gamma \geq 1$ does not look very promising at the moment.

As our main result, we present an approximation algorithm with performance ratio $\frac{1}{1-\frac{1}{2}(\gamma+\gamma^3)}$. This improves the result by Chandran and Ram for $\gamma \in [0.6507, 1)$, that is, for the particularly interesting large values of γ. The running time of our algorithm is $O(n^3)$, which matches the running time of the algorithm by Chandran and Ram.

1.2 Notations and conventions

For a set of nodes V, let $K(V)$ denote the set of edges $V \times V \setminus \{(v,v) \mid v \in V\}$. Throughout this work, we are considering directed graphs $G = (V, K(V))$

together with a weight function $w : K(V) \to \mathbb{Q}_{\geq 0}$ and a parameter $\gamma \in [\frac{1}{2}, 1)$. We always require that w fulfills the γ-parameterized triangle inequality (2). (Note that if w fulfills the γ-parameterized triangle inequality for some γ, then necessarily $\gamma \geq \frac{1}{2}$. Thus the lower bound is no restriction. We require $\gamma < 1$, since we already do not know how to achieve constant performance ratio for $\gamma = 1$.)

For a directed edge $e = (u, v)$, u is called the *tail* of e and v is called the *head* of e.

A cycle cover of a directed graph G is a spanning subgraph that consists solely of node disjoint directed cycles. A cycle is called a k-cycle if it has length *exactly* k. For any subgraph $S = (V, E)$ of G, the weight $w(S)$ of S is defined as the sum of the weights of the edges in E, that is, $w(S) = \sum_{e \in E} w(e)$. In particular, this defines the weight of cycle covers and TSP tours.

For a given directed graph G with weight function w, let $\mathrm{AB}(G)$ denote the weight of a minimum weight cycle cover. (This is also called the *assignment bound.*) Furthermore, let $\mathrm{TSP}(G)$ denote the weight of a minimum weight TSP tour of G. Obviously, we have $\mathrm{AB}(G) \leq \mathrm{TSP}(G)$. $\mathrm{AB}(G)$ and a corresponding minimum weight cycle cover can be computed in polynomial time.

2 Approximation algorithm

Figure 1 shows our new approximation algorithm. It generalizes a repeated cycle cover approach by Frieze, Galbiati, and Maffioli [8]. We first compute a minimum weight cycle cover C. This can be done in time $O(n^3)$. (There are various algorithms with this time bound, see e.g. [1] for an overview.) Then we choose from each cycle two nodes as representatives. One of them is placed in the set V_1, the other is put into the set V_2. Then we recursively compute two TSP tours T_1 and T_2, one in the graph G_1 induced by V_1 (i.e., $G_1 = (V_1, K(V_1))$ with weight function w_1 where w_1 is the restriction of w to $K(V_1)$) and the other in the graph G_2 induced by V_2. Then we combine C and the lighter of the two tours T_1 and T_2 and obtain a tour T by taking shortcuts.

The next lemma bounds the weight of a minimum weight TSP tour of G_1 and G_2 in terms of the weight of a minimum weight TSP tour of G.

Lemma 1. *Let C be a cycle cover of G and let $V_1, V_2 \subseteq V$ be disjoint sets such that V_1 and V_2 contain exactly one node from each cycle of C. Let G_1 and G_2 be the graphs induced by V_1 and V_2, respectively. Then*

$$\mathrm{TSP}(G_1) + \mathrm{TSP}(G_2) \leq (1 + \gamma^2) \cdot \mathrm{TSP}(G).$$

Proof. First we assume that each cycle in C is a 2-cycle. Thereafter, we reduce the general case to this special case.

Let T be a minimum weight TSP tour of G. Thus $w(T) = \mathrm{TSP}(G)$. We construct two TSP tours T_1 and T_2 of G_1 and G_2, respectively, such that $w(T_1) + w(T_2) \leq (1 + \gamma^2) \cdot w(T)$. This proves the claim of the lemma for the special case where each cycle of C is a 2-cycle.

Input: directed graph $G = (V, K(V))$ with weight function w where w fulfills the γ-parameterized triangle inequality (2) for some $\frac{1}{2} \leq \gamma < 1$.
Output: TSP tour T.

1. Compute a minimum weight cycle cover C of G.
2. Choose two disjoint sets $V_1, V_2 \subseteq V$ such that both V_1 and V_2 contain exactly one node from each cycle of C.
3. If $|V_1| > 1$ then recursively compute two TSP tours T_1 and T_2 of the graphs G_1 and G_2 that are induced by V_1 and V_2.
4. W.l.o.g. assume that T_1 is lighter than T_2. Construct T from C and T_1 as follows:
   (a) Construct an Eulerian tour E of $(V, C \cup T_1)$. This tour visits the nodes of V_1 in the order given by T_1. For each such node $v \in V_1$, the tour runs through the (unique) cycle in C that v_1 belongs to. After that, it goes on with the next node of T_1.
   (b) From E, we obtain T by taking *shortcuts*: Whenever E would visit a node it has visited before, T goes directly to the next node not visited before.

Fig. 1. Approximation algorithm for $\Delta(\gamma)$-ATSP

Given T, we construct T_1 and T_2 by taking *shortcuts*, that is, we move along the tour T starting with an arbitrary node in V_1 or V_2, respectively. Whenever we would visit a node not in V_1 or V_2, respectively, we directly go to the next node of T that is in V_1 or V_2.

Let $e = (u, v)$ be an edge of T. Since C consists solely of 2-cycles by assumption, $u, v \in V_1 \cup V_2$. If both u and v belong to V_1, then the edge e appears in T_1 but is contracted twice when constructing T_2. Since w satisfies the γ-parameterized triangle inequality, e contributes weight $w(e)$ to T_1 and $\gamma^2 \cdot w(e)$ to T_2 yielding a total contribution of $(1 + \gamma^2) \cdot w(e)$. If both u and v belong to V_2, the same analysis works. If u belongs to V_1 and v belongs to V_2 or vice versa, then e is contracted once to obtain T_1 and once to obtain T_2. Thus the total contribution is $2\gamma \cdot w(e) \leq (1 + \gamma^2) \cdot w(e)$. Summing over all edges e of T yields the result.

The special case proven above implies the general case as follows: We construct a TSP tour T' from T by taking shortcuts such that T' only visits nodes from V_1 and V_2. Since w particularly obeys the triangle inequality, $w(T') \leq w(T)$. Now we can apply the above special case to T'. $\square$

The following lemma bounds the weight of the final tour T in terms of the weight of C and T_1.

Lemma 2. *For the TSP tour T constructed in the algorithm in Figure 1, we have $w(T) \leq w(C) + \gamma \cdot w(T_1)$.*

Proof. The only nodes that are visited more than once (namely twice) in the Eulerian tour E are the nodes of T_1. Hence each edge e of T_1 is contracted once and yields weight only $\gamma \cdot w(e)$ in T. Thus the total weight of the constructed TSP tour T is at most $w(C) + \gamma \cdot w(T_1)$. (Also some edges of C are contracted. It is not clear how to get some improvement out of this observation, since these edges could have negligible weight.) $\square$

Now we can estimate the approximation performance of our algorithm.

Theorem 1. *The approximation performance of the algorithm in Figure 1 is bounded by* $\frac{1}{1-\frac{1}{2}(\gamma+\gamma^3)}$. *The running time of the algorithm is polynomial.*

Proof. The bound on the approximation performance is shown by induction in the number of nodes. For graphs with one node, the problem is trivial.

Suppose that G has more nodes. By Lemma 1, $\mathrm{TSP}(G_1) + \mathrm{TSP}(G_2) \le (1 + \gamma^2) \cdot \mathrm{TSP}(G)$. Particularly, $\mathrm{TSP}(G_i) \le \frac{1}{2}(1 + \gamma^2) \cdot \mathrm{TSP}(G)$ for some $i \in \{1, 2\}$. W.l.o.g. assume that $i = 1$.

By the induction hypothesis,

$$w(T_1) \le \frac{1}{1 - \frac{1}{2}(\gamma + \gamma^3)} \cdot \mathrm{TSP}(G_1) \le \frac{1 + \gamma^2}{2 - (\gamma + \gamma^3)} \cdot \mathrm{TSP}(G). \tag{3}$$

Furthermore we can assume that T_1 is indeed the lighter of the two TSP tours, because otherwise (3) also holds for T_2, as $w(T_2) \le w(T_1)$.

The TSP tour T computed in step 4 from C and T_1 has weight at most

$$\begin{aligned} w(C) + \gamma \cdot w(T_1) &\le \left(1 + \gamma \cdot \frac{1 + \gamma^2}{2 - (\gamma + \gamma^3)}\right) \cdot \mathrm{TSP}(G) \\ &\le \frac{2}{2 - (\gamma + \gamma^3)} \cdot \mathrm{TSP}(G). \end{aligned}$$

by Lemma 2. This proves the claim about the approximation performance.

Let $S(n)$ denote the worst case running time of the algorithm on instances with n nodes. We have $S(1) = 1$ and

$$S(n) \le 2 \cdot S(n/2) + O(n^3) \qquad \text{for all } n > 1,$$

because each instance is divided into two subproblems of size at most $n/2$. The time for computing the two subinstances is dominated by the time needed to construct the cycle cover C. Thus it is $O(n^3)$, see e.g. [1]. Solving the recurrence, we obtain $S(n) = O(n^3)$. $\square$

The approximation performance shown in Theorem 1 is better than the one obtained by Chandran and Ram [6], if

$$\frac{\gamma}{1-\gamma} \ge \frac{1}{1 - \frac{1}{2}(\gamma + \gamma^3)} \iff -\frac{\gamma^4}{2} - \frac{\gamma^2}{2} + 2\gamma - 1 \ge 0.$$

The real valued roots of the polynomial $p(\gamma) := -\frac{\gamma^4}{2} - \frac{\gamma^2}{2} + 2\gamma - 1$ can be computed exactly. They are $-\frac{1}{3} - \frac{5^{2/3}}{3(7+3\sqrt{6})^{1/3}} + \frac{1}{3}(5(7+3\sqrt{6}))^{1/3}$ and 1. In particular, $p(\gamma) > 0$ holds for $\gamma \in [0.6507, 1)$. Figure 2 compares the two performances in dependence of γ.

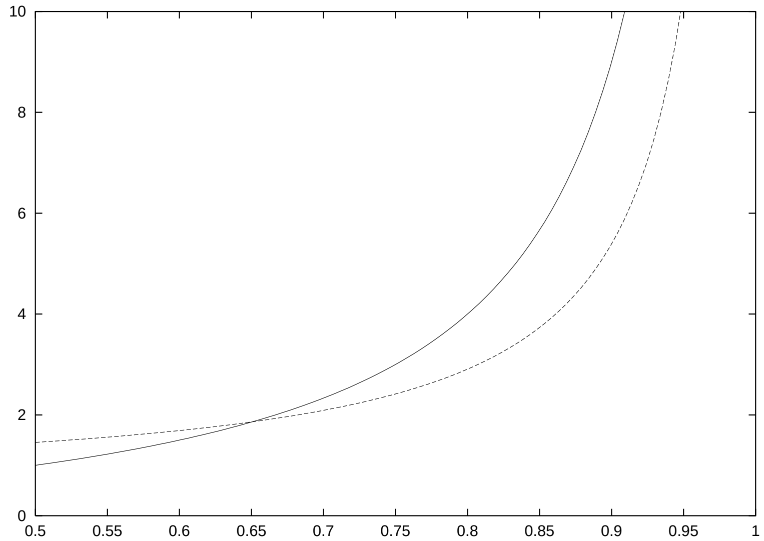

Fig. 2. The approximation performance of the algorithm in Figure 1 (drawn dashed) compared to the one by Chandran and Ram (drawn solid).

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