Depth reduction in arithmetic circuits

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$$\begin{aligned} f(x_1,x_2,x_3,x_4) \quad = \quad & 1+x_1+x_2+x_3+x_4 \\ & +x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4 \\ & +x_2x_3x_4+x_1x_3x_4+x_1x_2x_4+x_1x_2x_3 \\ & +x_1x_2x_3x_4 \end{aligned}$$

Polynomials

$$
\begin{aligned}
f(x_1, x_2, x_3, x_4) \quad &= \quad 1 + x_1 + x_2 + x_3 + x_4 \\
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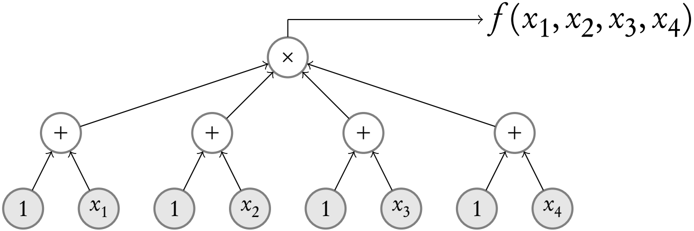
&\quad + x_1x_2x_3x_4 \\
\\
&= \quad (1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4)
\end{aligned}
$$

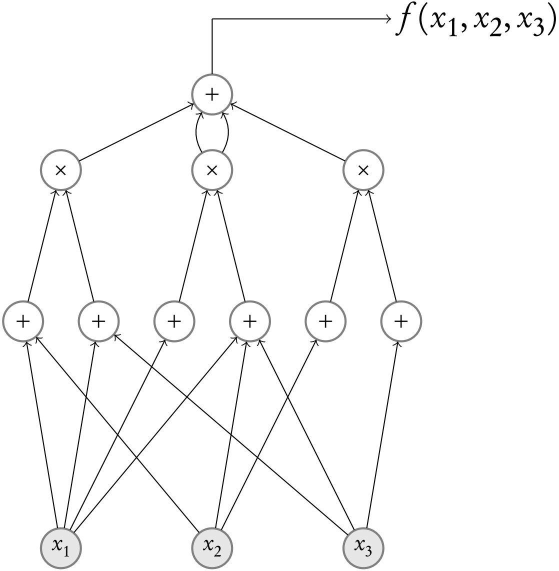
"How hard is it to compute a given n-variate degree d polynomial?"

Arithmetic Formulae

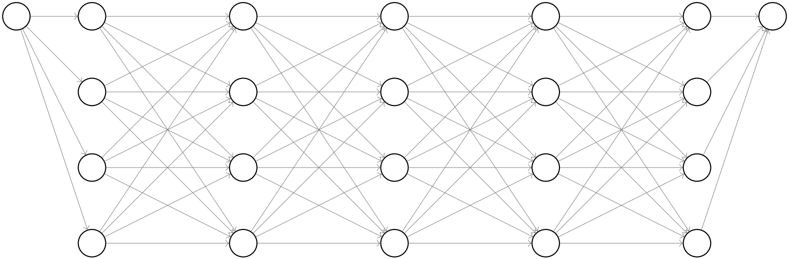


- Tree
- Leaves containing variables or constants

Arithmetic Circuits



Arithmetic Branching Programs



A directed layered graph with a unique source node s and sink node t.
Each edge e holds a linear polynomial ℓ_e.

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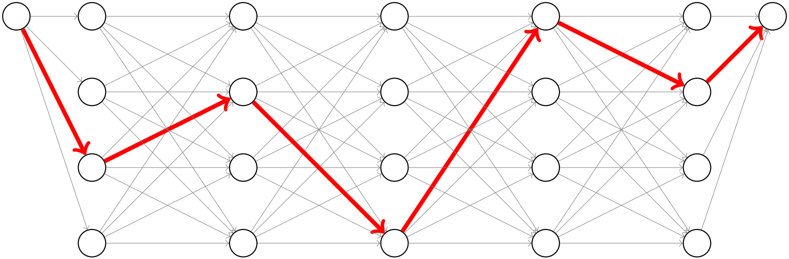

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$$\mathrm{wt}(P) \quad = \quad \prod_{e \in P} \ell_e$$

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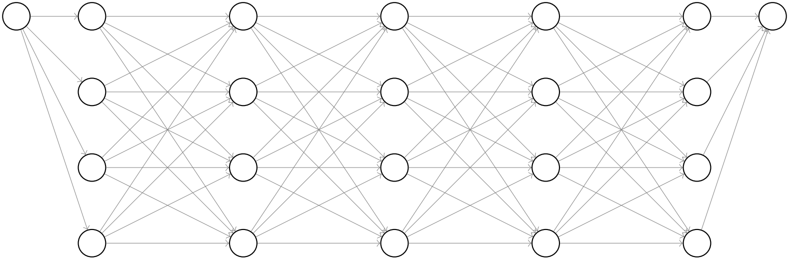

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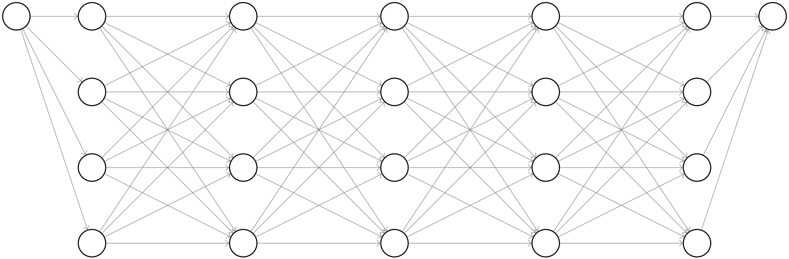

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Equivalent to iterated matrix product

Relationship between these classes

Formulas $\subseteq$ ABP

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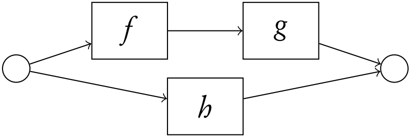
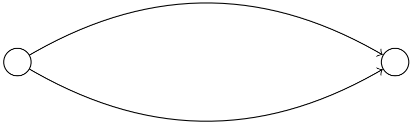
$f\,g + h$:



Relationship between these classes

Formulas $\subseteq$ ABP $\subseteq$ Circuits



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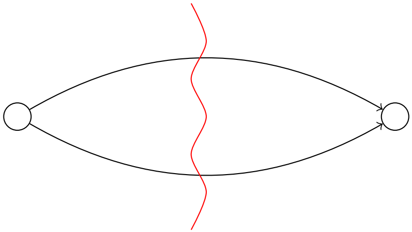

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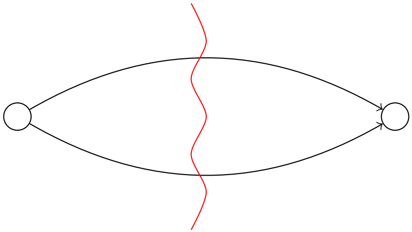

Savitch

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$$g = g_1 + g_2 \quad \longrightarrow \quad g^{(i)} = g_1^{(i)} + g_2^{(i)}$$

$$g = g_1 \times g_2 \quad \longrightarrow \quad g^{(i)} = \sum_{j=0}^{i} g_1^{(j)} \times g_2^{(i-j)}$$

The illustrious siblings

$$\mathrm{Det}_n(x_{11},\ldots,x_{nn}) = \sum_{\sigma\in S_n} \mathrm{sign}(\sigma)\cdot x_{1\sigma(1)}\ldots x_{n\sigma(n)}$$

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Det_n		VP
vs	[Valiant-79] ⟷	vs
Perm_n		VNP

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- We would obtain new algorithms for polynomial identity testing.
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General roadmap for lower bounds

Four steps in most lower bound proofs

Step 1: Finding the right building blocks

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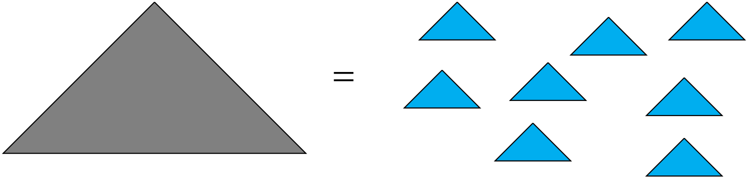

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Every small circuit can be equivalently computed as a *sum* of few s

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Step 2: Constructing a complexity measure

Meta Theorem 2

Find a map $\Gamma : \mathbb{F}[\mathbf{x}] \to \mathbb{Z}_{\geq 0}$ such that Γ () is small.

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Step 3: Heuristic estimate for a random polynomial

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Convince yourself that $\Gamma(R)$ *must be* LARGE for a random polynomial R.

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Step 4: Find a hay in the haystack

Four steps in most lower bound proofs

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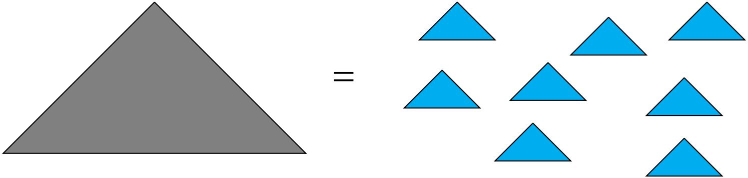

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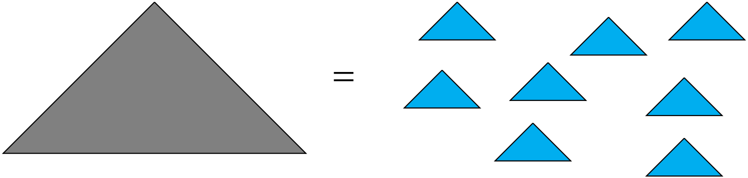

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Depth reduction

A short history of depth reduction

Circuit Class	Depth	Size	
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Circuits	3^*	$s^{O(\sqrt{d})}$	[Gupta-Kamath-Kayal-S]

Other depth reductions in lower bounds

Multilinear formulas [Raz, Raz-Yehudayoff]

$$f \quad = \quad \sum_{i=1}^{s} g_{i1} \cdot g_{i2} \dots g_{i\ell} \quad , \quad (1/3)^j \cdot n \leq \mathrm{Var}(g_{ij}) \leq (2/3)^j \cdot n$$

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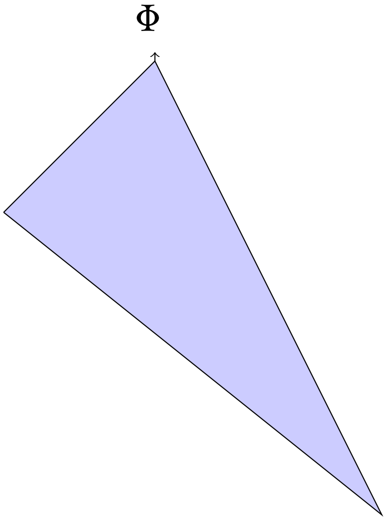
Homogeneous $\Sigma\Pi\Sigma\Pi$ [Kayal-Limaye-Saha-Srinivasan]

$$f = \Sigma\Pi\Sigma\Pi^{[\sqrt{d}]} + \sum_{i=1}^{s} m_i Q_i \quad , \quad \deg(m_i) \geq \sqrt{d}$$

Plan

- Classical depth reductions of [Brent] and [VSBR].
- A *slightly* different proof of [Tavenas]
- Better building blocks for homogeneous formulas
- (depending on time) Reduction to depth three [GKKS2]

Depth reducing formulas



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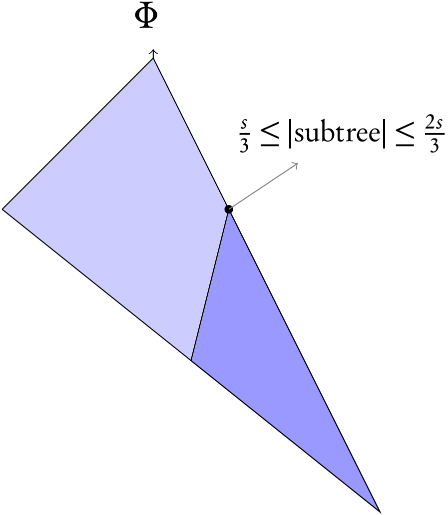

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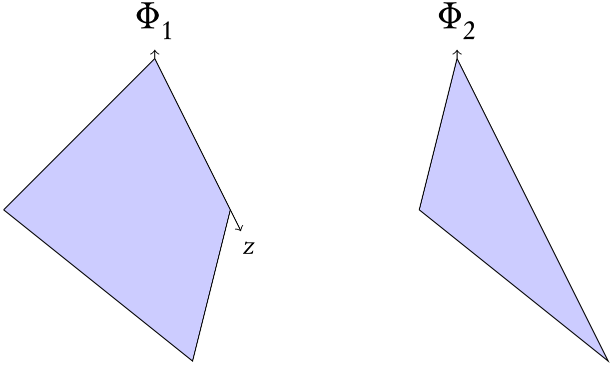

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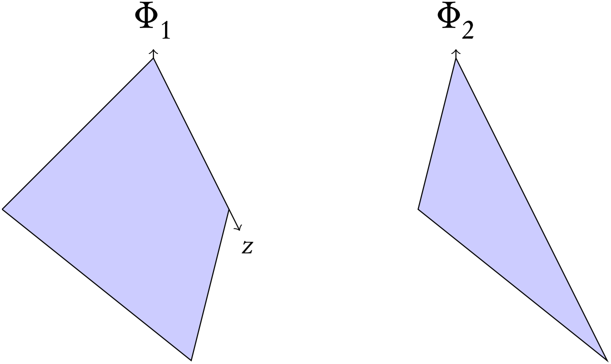

$$\Phi_1(z) \quad = \quad A \cdot z \quad + \quad B$$
$$\Phi \quad = \quad A \cdot \Phi_2 \quad + \quad B$$

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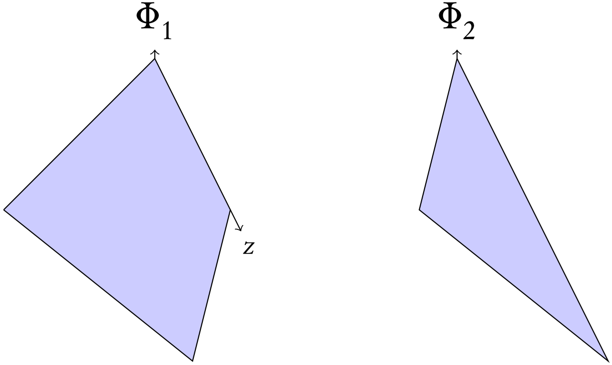

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$$\Phi = A \cdot \Phi_2 + B = (\Phi_1(1) - \Phi_1(0)) \cdot \Phi_2 + \Phi_1(0)$$

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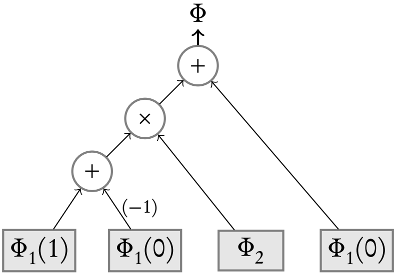

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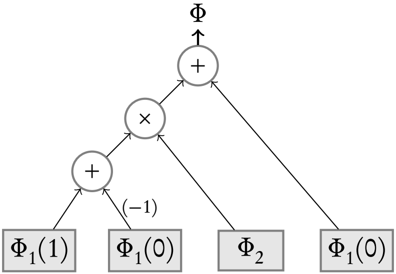
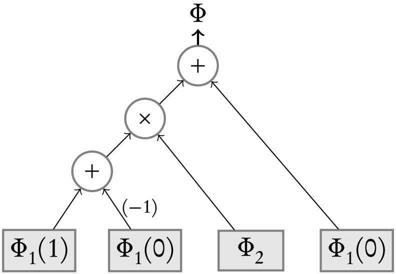


$$\mathrm{Size}(s) \leq 4 \cdot \mathrm{Size}(2s/3) + O(1)$$
$$\mathrm{Depth}(s) \leq \mathrm{Depth}(2s/3) + O(1)$$

Depth reducing formulas



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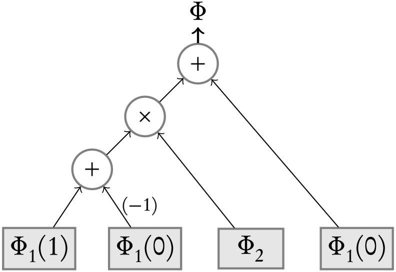

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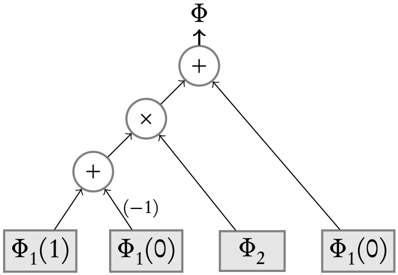

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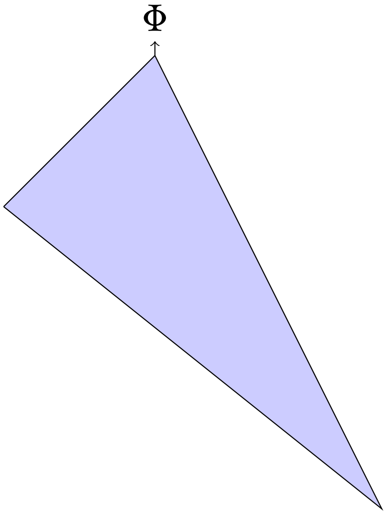
□

Adapting to circuits



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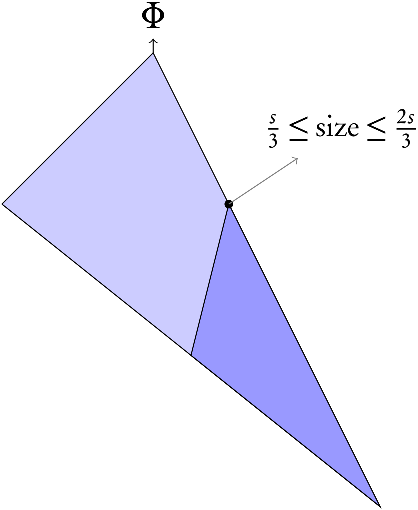

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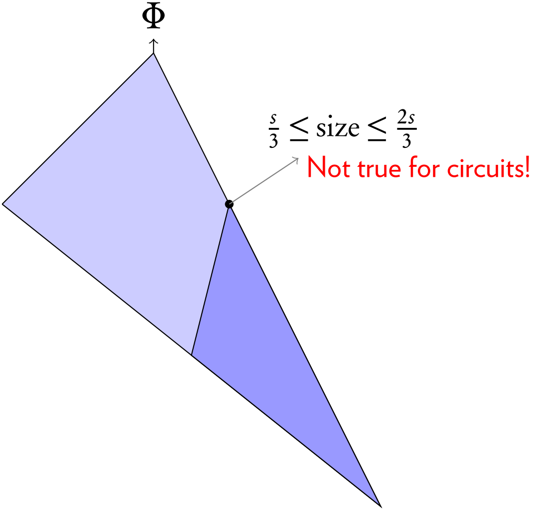

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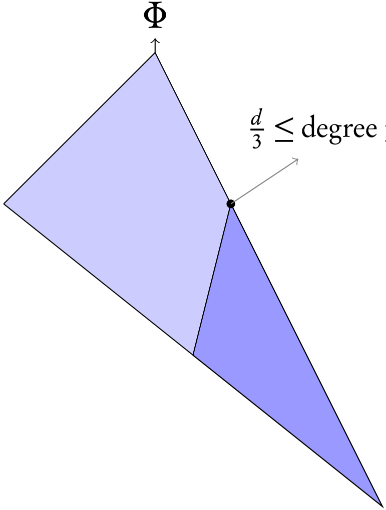

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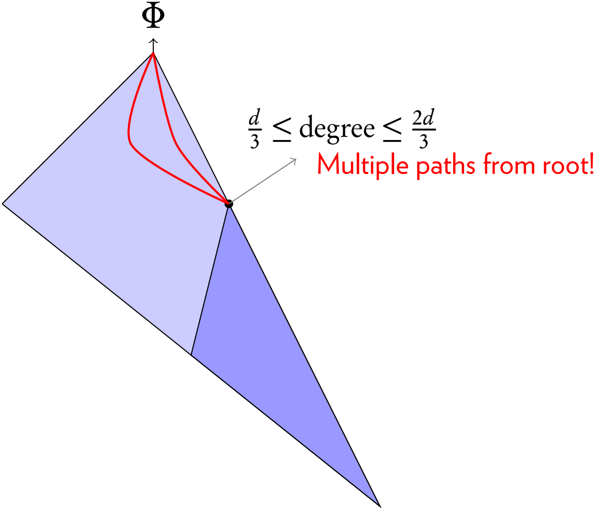

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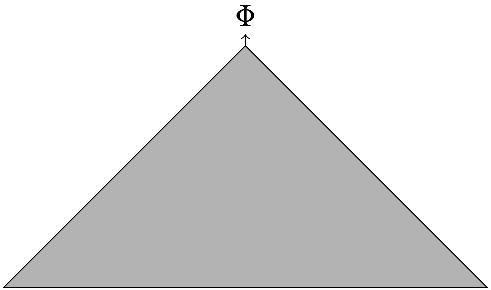

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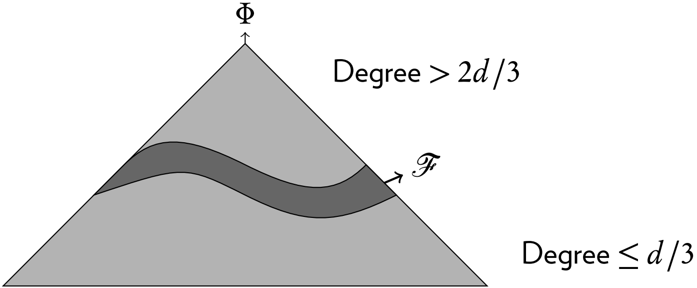

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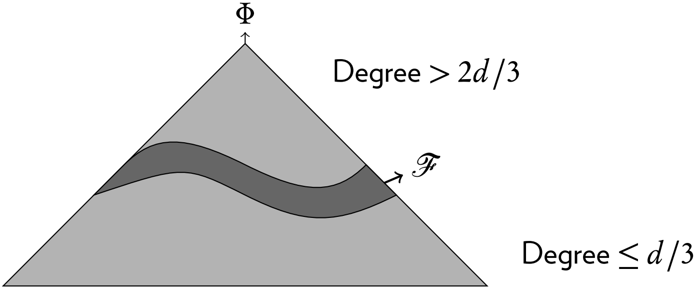

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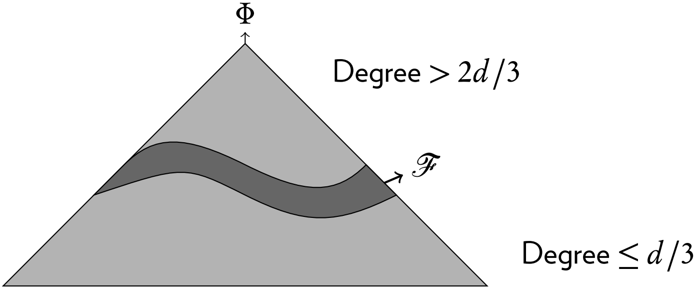

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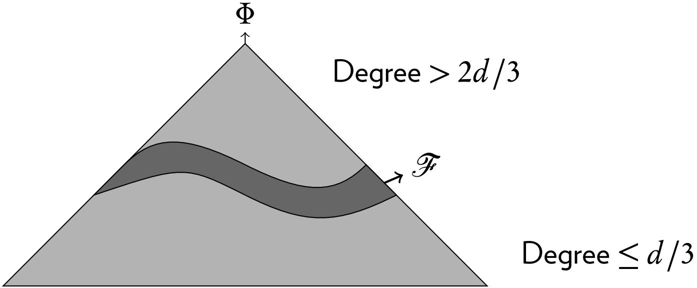

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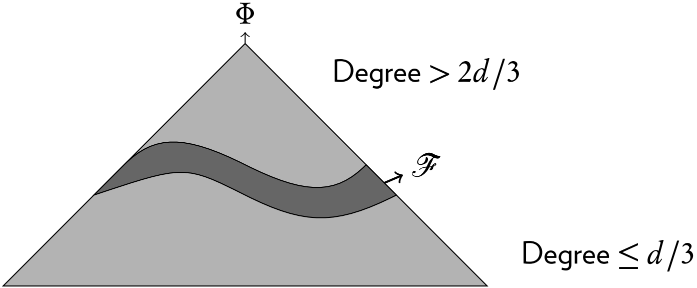
Interpolate!

Adapting to circuits: Attempt 1



$$\mathscr{F} = \left\{ v \in \Phi : \frac{d}{3} < \deg(v) \leq \frac{2d}{3} \right\}$$

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$$\text{Depth}(d) = \text{Depth}(2d/3) + O(1)$$

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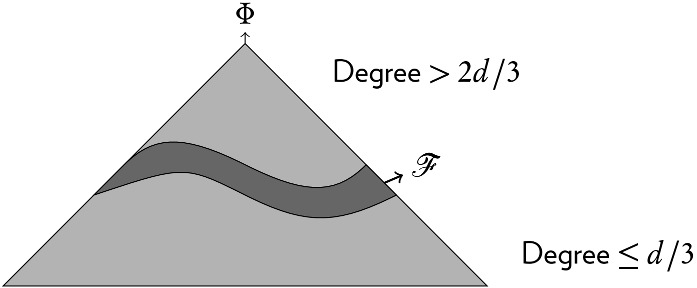

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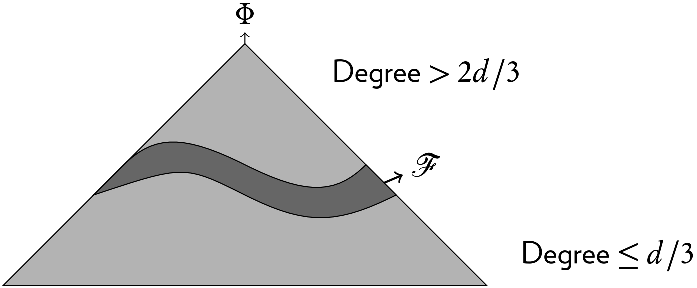

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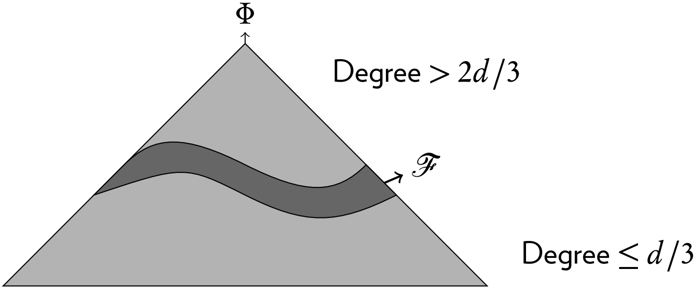

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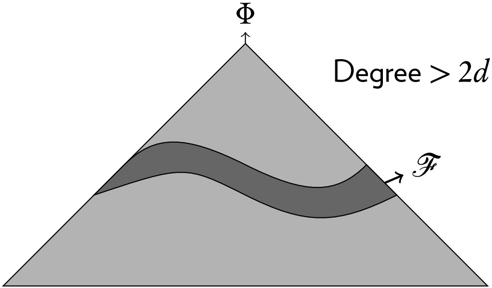
Adapting to circuits: [Hyafil]



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$$[u : v] \quad = \quad \begin{cases} 0 & \text{if } u \text{ is a leaf} \\ 1 & \text{if } u = v \\ [u_1 : v] + [u_2 : v] & \text{if } u = u_1 + u_2 \\ [u_1] \cdot [u_2 : v] & \text{if } u_1 = u_1 \times u_2 \end{cases}$$

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Another possibility: Partial Derivatives.

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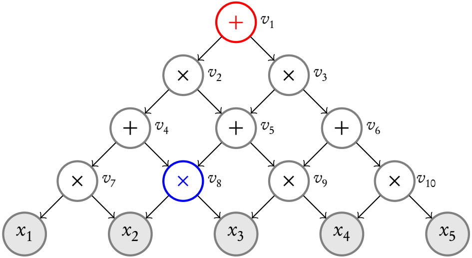
Another possibility: Partial Derivatives.
Works, but one needs to be a little careful with multiple paths. See [Shpilka-Yehudayoff]

An example



$$[v_1 : v_8] \quad =$$

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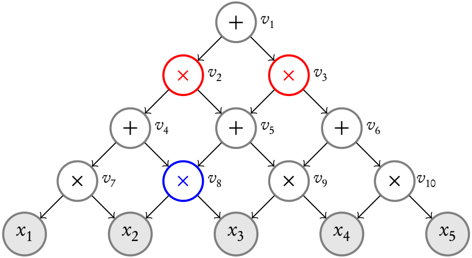

$$[v_1 : v_8] = [v_2 : v_8] + [v_3 : v_8]$$

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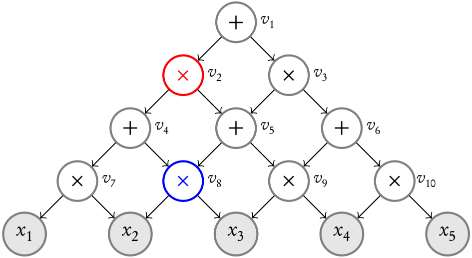

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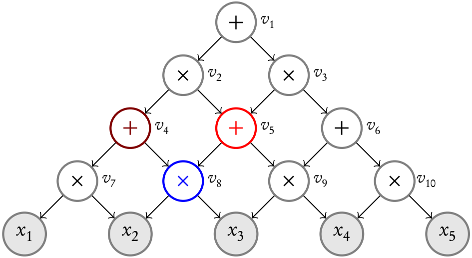

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[v_1 : v_8] &= [v_2 : v_8] + \cancel{[v_3 : v_8]} \\
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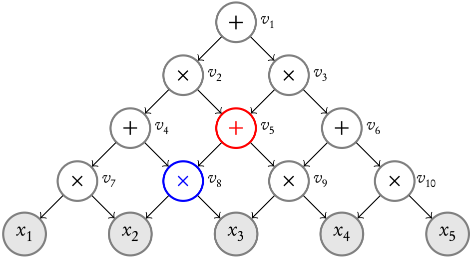

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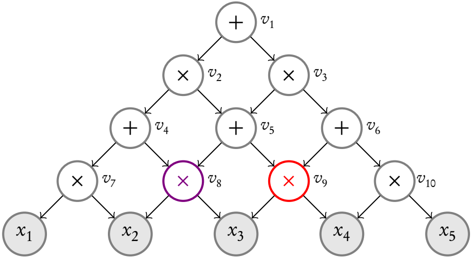

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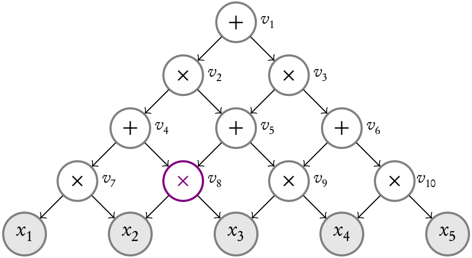

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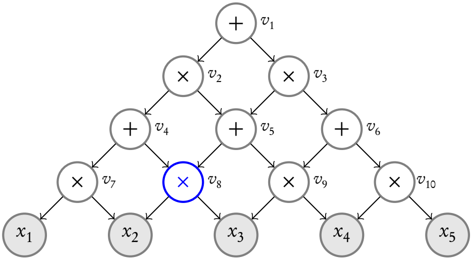

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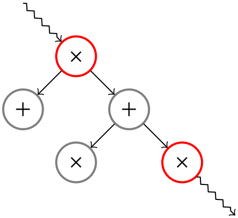

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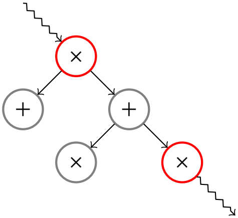

[VSBR] *continued ...*

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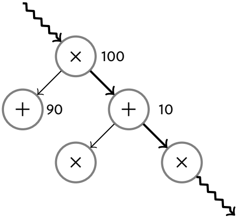

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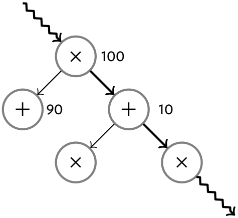

Make the circuit *right heavy*.

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Summarizing

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Theorem ([Valiant-Skyum-Berkowitz-Rackoff])

If Φ is a size s circuit computing an n-variate degree d polynomial f, then there is a circuit Φ' computing f with the following properties.

- *Every gate of Φ' computes some $[u:v]$,*
- *All addition gates have fan-in at most s^2,*
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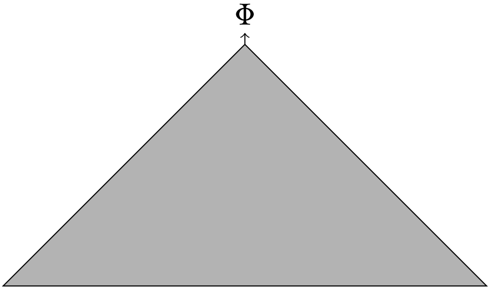
- *If v_1 is a child of a $\times$-gate v in Φ', then $\deg(v_1) \leq \deg(v)/2$. Hence, the depth of Φ' is $O(\log d)$.*

Reducing to depth four



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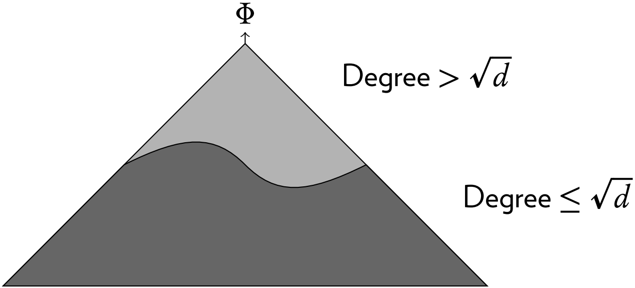

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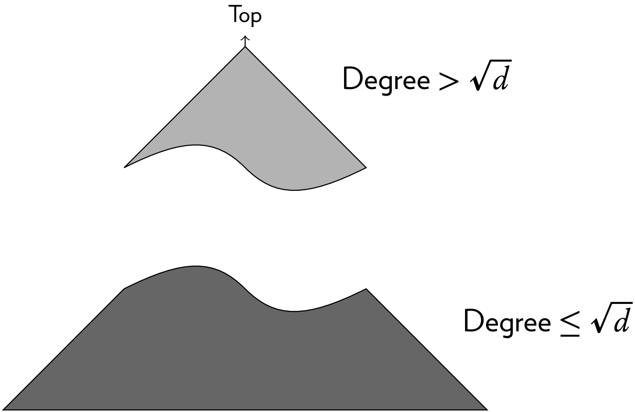
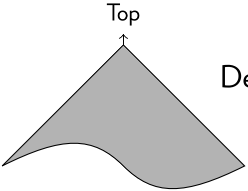
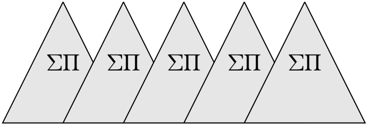


Reducing to depth four



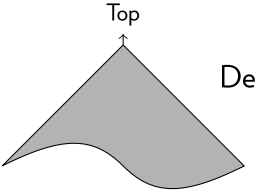
Degree $> \sqrt{d}$



Degree $\leq \sqrt{d}$

Size $\binom{n+\sqrt{d}}{\sqrt{d}}$ each

Reducing to depth four



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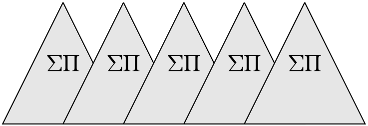

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Lemma ([Tavenas13])

$$\deg(\mathrm{Top}(z_1, \ldots, z_s)) \quad \leq \quad 15\sqrt{d}$$

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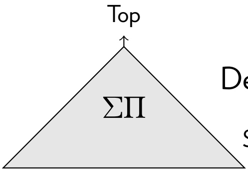
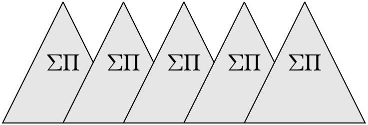


Degree $> \sqrt{d}$

Size $\binom{s+15\sqrt{d}}{15\sqrt{d}}$



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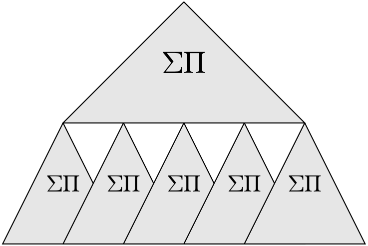
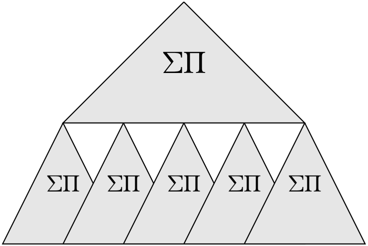


Theorem

Equivalent depth-4 circuit of size

$$s\binom{n+\sqrt{d}}{n} \quad + \quad \binom{s+15\sqrt{d}}{s} \quad = \quad s^{O(\sqrt{d})}$$

Reducing to depth four



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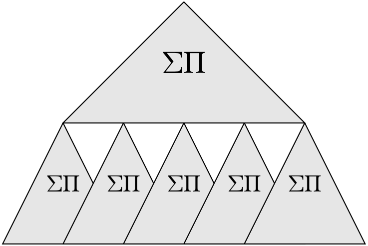


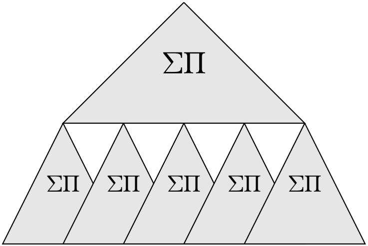
Theorem

Equivalent depth-4 circuit with bottom fan-in at most $\sqrt{d}$ of size

$$s\binom{n+\sqrt{d}}{n} \quad + \quad \binom{s+15\sqrt{d}}{s} \quad = \quad s^{O(\sqrt{d})}$$

□

Reducing to depth four



Theorem

Equivalent $\Sigma\Pi\Sigma\Pi^{[\sqrt{d}]}$ *circuit of size*

$$s\binom{n+\sqrt{d}}{n} + \binom{s+15\sqrt{d}}{s} = s^{O(\sqrt{d})}$$

□

A different perspective

Let's start with [VSBR]

$$f \quad = \quad \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}$$

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A different perspective

Let's start with [VSBR]

$$f = \sum_{i=1}^{s^3} f_{i1} \cdots f_{i13}$$

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Let's start with [VSBR]

$$f = \sum_{i=1}^{s^4} f_{i1} \cdots f_{i17}$$

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How many iterations until all degrees are at most t?

Number of iterations

$$g = \sum_{j=1}^{s} g_{j1} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

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Observation

In each summand, at least two terms have degree at least $t/8$.

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$$g = \sum_{j=1}^{s} \underbrace{g_{j1}}_{\geq t/5} \cdot g_{j2} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

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How many factors of degree at least $t/8$?

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Number of iterations

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How many factors of degree at least $t/8$?

$$f = \sum_{i=1}^{s^2} f_{i1} \cdot f_{i12} \cdot f_{i3} \cdot f_{i4} \cdots f_{i9}$$

Number of iterations

$$g = \sum_{j=1}^{s} \underbrace{g_{j1}}_{\geq t/5} \cdot \underbrace{g_{j2}}_{\geq t/8} \cdot g_{j3} \cdot g_{j4} \cdot g_{j5}$$

Observation

In each summand, at least two terms have degree at least $t/8$.

How many factors of degree at least $t/8$? At most $8d/t$.

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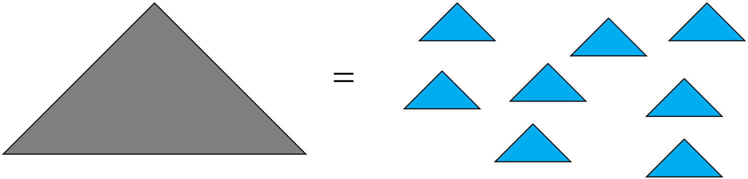
Final $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit has top fan-in at most $s^{O(d/t)}$. □

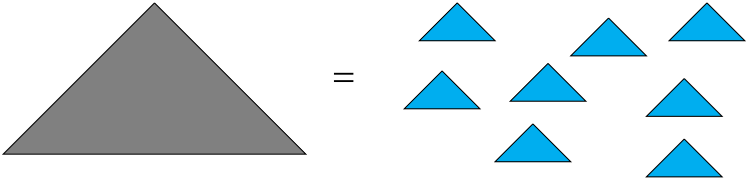
Recap



Theorem

Every small circuit can be equivalently computed as a *sum* of few s

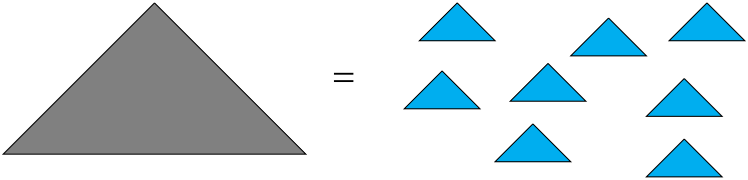
Recap



Theorem

Every circuit of size s can be equivalently computed as a *sum* of few s

Recap

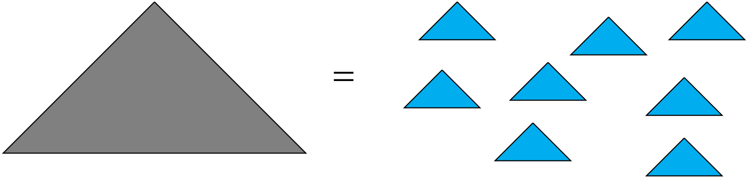


Theorem

Every circuit of size s can be equivalently computed as a *sum* of $s^{O(d/t)}$

s

Recap

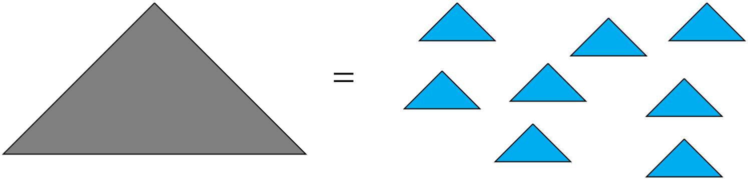


Theorem

Every circuit of size s can be equivalently computed as a *sum* of $s^{O(d/t)}$ s , where

 $$= \prod_{i=1}^{O(d/t)} Q_i \qquad \deg(Q_i) \leq t$$

Recap



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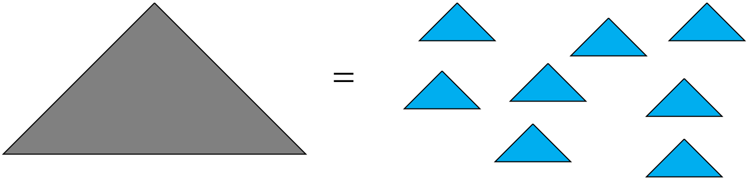
What are the s if we start with a homogeneous formula of size s?

Recap



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What are the s if we start with a depth 100 formula of size s?

A better starting point?

$$f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdot f_{i3} \cdot f_{i4} \cdot f_{i5}$$

If we start with a homogeneous formula, can we do better?

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[Hrubes-Yehudayoff]: Yes!

Lemma ([Hrubes-Yehudayoff])

$$f \quad = \quad \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdots f_{i\ell} \qquad \textit{with } \left(\frac{1}{3}\right)^{j} \cdot d < \deg(f_{ij}) \leq \left(\frac{2}{3}\right)^{j} \cdot d$$

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Proof

$$f = A \cdot \Phi_v + \Phi_{v=0}$$

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Proof

$$\begin{aligned} f \quad &= \quad A \cdot \Phi_v + \Phi_{v=0} \\ &= \quad A \cdot \left(\sum_{i=1}^{s_1} g_{i1} \cdots g_{i\ell} \right) + \left(\sum_{j=1}^{s_2} h_{j1} \cdots h_{j\ell} \right) \end{aligned}$$

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Reduction to depth four, again

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Fact: Each f_{ij} is also computable by size s homogeneous formulas

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Question: How many iterations? $O(d/t)$ again.

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Proof: There are at least two terms of degree $t/9$. Yada Yada Yada □

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Yields a $\Sigma\Pi\Sigma\Pi^{[t]}$ circuit of top fan-in $s^{O(d/t)}$.

Wait... what's different?

For circuits:

$$f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdots f_{i5}$$

For homogeneous formulas:

$$f = \sum_{i=1}^{s} f_{i1} \cdot f_{i2} \cdots f_{i\ell}$$

Wait... what's different?

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$$f = \sum_{i=1}^{s} \left(\sum_{j=1}^{s} g_{i1} \cdots g_{i5} \right) \cdot f_{i2} \cdots f_{i5}$$

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Wait... what's different?

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$$f = \sum_{i=1}^{s^2} f_{i1} \cdot f_{i2} \cdots f_{i9}$$

For homogeneous formulas:

$$f = \sum_{i=1}^{s^2} f_{i1} \cdot f_{i2} \cdots f_{i(2\ell)}$$

Wait... what's different?

For circuits:

$$f \quad = \quad \sum_{i=1}^{s^4} f_{i1} \cdot f_{i2} \cdots f_{i13}$$

For homogeneous formulas:

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For circuits:

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a $\Sigma\Pi^{[O(d/t)]}\Sigma\Pi^{[t]}$ circuit

For homogeneous formulas:

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Here, $\triangle$s factorize more

Is that a big deal?

Circuit class	No. factors of 	Lower bound

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Hom. $\Sigma\Pi\Sigma$	$O(n)$	$n^{\Omega(n)}$

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$\Sigma\Pi\Sigma\Pi^{[t]}$	$O(d/t)$	$n^{\Omega(d/t)}$

Is that a big deal?

Circuit class	No. factors of	Lower bound
Hom. $\Sigma\Pi\Sigma$	$O(n)$	$n^{\Omega(n)}$
Multilinear formulas	$O(\log n)$	$n^{\Omega(\log n)}$
Multilinear, depth-Δ	$n^{O(1/\Delta)}$	$n^{n^{\Omega(1/\Delta)}}$
$\Sigma\Pi\Sigma\Pi^{[t]}$	$O(d/t)$	$n^{\Omega(d/t)}$
Hom. $\Sigma\Pi\Sigma\Pi$	$O(\sqrt{d})$	$n^{O(\sqrt{d})}$

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Hom. $\Sigma\Pi\Sigma\Pi$	$O(\sqrt{d})$	$n^{O(\sqrt{d})}$

Wishful Question: Can we get an $n^{\Omega(\log n)}$ lower bound for homogeneous formulas, using current techniques (with slight modifications)?

Reduction to Depth-3 Circuits

No time!

Road map

$$\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$$

circuits

$$\downarrow$$

$$\Sigma\wedge^{\sqrt{d}}\Sigma\wedge^{\sqrt{d}}\Sigma$$

circuits

$$\downarrow$$

$$\Sigma\Pi\Sigma$$

circuits

Road map

$\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ circuits

↓ App. of Ryser's formula

$\Sigma\bigwedge^{\sqrt{d}}\Sigma\bigwedge^{\sqrt{d}}\Sigma$ circuits

↓

$\Sigma\Pi\Sigma$ circuits

Road map

$$\Sigma \overset{\sqrt{d}}{\Pi} \Sigma \overset{\sqrt{d}}{\Pi}$$

circuits

↓ App. of Ryser's formula

$$\Sigma \overset{\sqrt{d}}{\bigwedge} \Sigma \overset{\sqrt{d}}{\bigwedge} \Sigma$$

circuits

↓ [Saxena]'s duality trick

$$\Sigma \Pi \Sigma$$

circuits

Road map

$$\Sigma\prod^{\sqrt{d}}\Sigma\prod^{\sqrt{d}}$$

circuits

↓

Only over $\mathbb{Q},\mathbb{R}$ etc. — App. of Ryser's formula

$$\Sigma\bigwedge^{\sqrt{d}}\Sigma\bigwedge^{\sqrt{d}}\Sigma$$

circuits

↓

Heavily non-homogeneous — [Saxena]'s duality trick

$$\Sigma\Pi\Sigma$$

circuits

Step 1: $\Sigma\Pi\Sigma\Pi$ *to* $\Sigma\wedge\Sigma\wedge\Sigma$

Step 1*:* $\Sigma\Pi\Sigma\Pi$ *to* $\Sigma\wedge\Sigma\wedge\Sigma$

Recall Ryser's formula:

$$\mathrm{Perm}_n \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} = \sum_{S \subseteq [n]} (-1)^{n-|S|} \prod_{i=1}^{n} \sum_{j \in S} x_{ij}$$

Step 1*:* $\Sigma\Pi\Sigma\Pi$ *to* $\Sigma\wedge\Sigma\wedge\Sigma$

Recall Ryser's formula:

$$\mathrm{Perm}_n \begin{bmatrix} x_1 & \cdots & x_n \\ \vdots & \ddots & \vdots \\ x_1 & \cdots & x_n \end{bmatrix} = \sum_{S\subseteq[n]} (-1)^{n-|S|} \prod_{i=1}^{n} \sum_{j\in S} x_j$$

Step 1*:* $\Sigma\Pi\Sigma\Pi$ *to* $\Sigma\wedge\Sigma\wedge\Sigma$

Recall Ryser's formula:

$$\mathrm{Perm}_n \begin{bmatrix} x_1 & \dots & x_n \\ \vdots & \ddots & \vdots \\ x_1 & \dots & x_n \end{bmatrix} = \sum_{S\subseteq[n]} (-1)^{n-|S|} \left(\sum_{j\in S} x_j \right)^n$$

Step 1*:* $\Sigma\Pi\Sigma\Pi$ *to* $\Sigma\wedge\Sigma\wedge\Sigma$

Recall Ryser's formula:

$$n! \cdot x_1 \dots x_n \quad = \quad \sum_{S \subseteq [n]} (-1)^{n-|S|} \left(\sum_{j \in S} x_j \right)^n$$

Step 1*:* $\Sigma\Pi\Sigma\Pi$ *to* $\Sigma\wedge\Sigma\wedge\Sigma$

[Fischer]:

$$n! \cdot x_1 \ldots x_n \quad = \quad \sum_{S \subseteq [n]} (-1)^{n-|S|} \left(\sum_{j \in S} x_j \right)^n$$

Step 1: $\Sigma\Pi\Sigma\Pi$ *to* $\Sigma\wedge\Sigma\wedge\Sigma$

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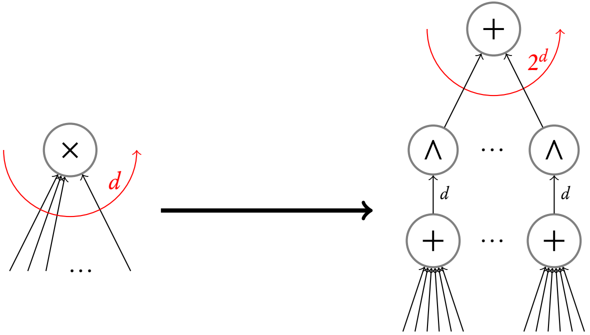
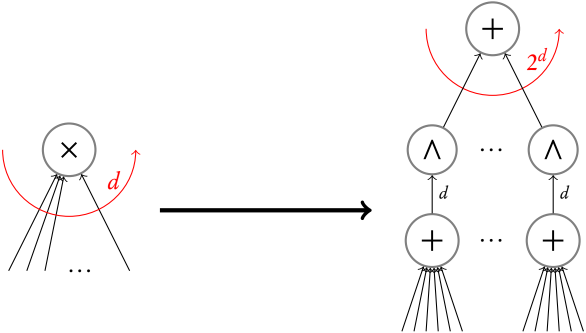
$$n! \cdot x_1 \dots x_n \quad = \quad \sum_{S \subseteq [n]} (-1)^{n-|S|} \left(\sum_{j \in S} x_j \right)^n$$

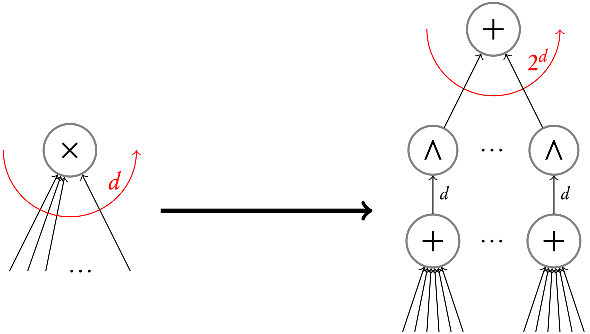


Step 1: $\Sigma\Pi\Sigma\Pi$ *to* $\Sigma\wedge\Sigma\wedge\Sigma$



$$\overset{d}{\prod} \longrightarrow \overset{2^d}{\Sigma}\overset{d}{\wedge}\overset{d}{\Sigma}$$

Step 1: $\Sigma\Pi\Sigma\Pi$ *to* $\Sigma\Lambda\Sigma\Lambda\Sigma$



$$\prod^{d} \longrightarrow \sum^{2^d}\bigwedge^{d}\sum^{d}$$

$$\sum\prod^{\sqrt{d}}\sum\prod^{\sqrt{d}} \text{ of size } s \longrightarrow \sum\bigwedge^{\sqrt{d}}\sum\bigwedge^{\sqrt{d}}\sum \text{ of size } 2^{O(\sqrt{d})}\cdot s$$

Road map

$$\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$$

circuits

↓

$$\Sigma\wedge^{\sqrt{d}}\Sigma\wedge^{\sqrt{d}}\Sigma$$

circuits

↓

$$\Sigma\Pi\Sigma$$

circuits

Road map

$$\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$$

circuits

✓

$$\Sigma\wedge^{\sqrt{d}}\Sigma\wedge^{\sqrt{d}}\Sigma$$

circuits

$$\Sigma\Pi\Sigma$$

circuits

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$\Sigma \overset{\sqrt{d}}{\wedge} \Sigma \overset{\sqrt{d}}{\wedge} \Sigma$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$\Sigma \overset{\sqrt{d}}{\wedge} \Sigma \overset{\sqrt{d}}{\wedge} \Sigma$$

$$\ell$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$\Sigma \overset{\sqrt{d}}{\wedge} \Sigma \overset{\sqrt{d}}{\wedge} \Sigma$$

$$\ell^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$\Sigma \overset{\sqrt{d}}{\wedge} \Sigma \overset{\sqrt{d}}{\wedge} \Sigma$$

$$\ell_1^{\sqrt{d}} + \ldots + \ell_s^{\sqrt{d}}$$

Step 2: $\Sigma\Lambda\Sigma\Lambda\Sigma$ *to* $\Sigma\Pi\Sigma$

$$\Sigma \overset{\sqrt{d}}{\Lambda} \Sigma \overset{\sqrt{d}}{\Lambda} \Sigma$$

$$\left(\ell_1^{\sqrt{d}} + \ldots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$\Sigma \overset{\sqrt{d}}{\wedge} \Sigma \overset{\sqrt{d}}{\wedge} \Sigma$$

$$\sum_i \left(\ell_{i1}^{\sqrt{d}} + \ldots + \ell_{is}^{\sqrt{d}} \right)^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$C = \sum_i \left(\ell_{i1}^{\sqrt{d}} + \cdots + \ell_{is}^{\sqrt{d}}\right)^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

Lemma ([Saxena])

There exists univariate polynomials f_{ij}'s of degree at most d such that

$$(x_1 + \cdots + x_s)^d \quad = \quad \sum_{i=1}^{sd+1} \prod_{j=1}^{s} f_{ij}(x_j)$$

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Sketch of a proof by Shpilka

$$P_{\mathbf{x}}(t) = (1 + x_1 t)\ldots(1 + x_s t) = 1 + \ell t + \text{(higher degree terms)}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ to $\Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

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Sketch of a proof by Shpilka

$$P_{\mathbf{x}}(t) - 1 \quad = \quad \ell t \quad + \quad \text{(higher degree terms)}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ to $\Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

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Sketch of a proof by Shpilka

$$(P_{\mathbf{x}}(t) - 1)^d \quad = \quad \ell^d t^d \quad + \quad \text{(higher degree terms)}$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ to $\Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

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Sketch of a proof by Shpilka

$$(P_{\mathbf{x}}(t) - 1)^d \quad = \quad \ell^d t^d \quad + \quad \text{(higher degree terms)}$$

Interpolate!

$(P_{\mathbf{x}}(t) - 1)^d$ expanded is a sum of $(d+1)$ product of univariates. □

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$(x_1 + \cdots + x_s)^{\sqrt{d}} = \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} f_{ij}\left(x_j\right)$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} \quad = \quad \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} f_{ij}\left(\ell_j^{\sqrt{d}}\right)$$

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$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} \quad = \quad \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} f_{ij}\left(\ell_j^{\sqrt{d}}\right)$$

$$= \quad \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} \tilde{f}_{ij}(\ell_j)$$

where $\tilde{f}_{ij}(t) := f_{ij}(t^{\sqrt{d}})$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} \quad = \quad \sum_{i}^{\text{poly}(s,d)} \prod_{j=1}^{s} f_{ij}\left(\ell_j^{\sqrt{d}}\right)$$

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Note that $\tilde{f}_{ij}(t)$ is a univariate polynomial

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} = \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} f_{ij}\left(\ell_j^{\sqrt{d}}\right)$$

$$= \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} \tilde{f}_{ij}(\ell_j)$$

Note that $\tilde{f}_{ij}(t)$ is a univariate polynomial that can be factorized over $\mathbb{C}$:

$$\tilde{f}_{ij}(t) = \prod_{k=1}^{d}(t - \zeta_{ijk})$$

Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} \quad = \quad \sum_i^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} f_{ij}\left(\ell_j^{\sqrt{d}}\right)$$

$$= \quad \sum_i^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} \tilde{f}_{ij}(\ell_j)$$

Note that $\tilde{f}_{ij}(t)$ is a univariate polynomial that can be factorized over $\mathbb{C}$:

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Step 2: $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\begin{aligned}\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} &= \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} f_{ij}\left(\ell_j^{\sqrt{d}}\right) \\ &= \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} \tilde{f}_{ij}(\ell_j) \\ &= \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} \prod_{k=1}^{d} \left(\ell_j - \zeta_{ijk}\right)\end{aligned}$$

Step 2*:* $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T = \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\begin{aligned}
\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} &= \sum_i^{\text{poly}(s,d)} \prod_{j=1}^{s} f_{ij}\left(\ell_j^{\sqrt{d}}\right) \\
&= \sum_i^{\text{poly}(s,d)} \prod_{j=1}^{s} \tilde{f}_{ij}(\ell_j) \\
&= \sum_i^{\text{poly}(s,d)} \prod_{j=1}^{s} \prod_{k=1}^{d} \left(\ell_j - \zeta_{ijk}\right)
\end{aligned}$$

... a $\Sigma\Pi\Sigma$ circuit of poly(s,d) size.

Step 2*:* $\Sigma\wedge\Sigma\wedge\Sigma$ *to* $\Sigma\Pi\Sigma$

$$T \quad = \quad \left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}}$$

$$\left(\ell_1^{\sqrt{d}} + \cdots + \ell_s^{\sqrt{d}}\right)^{\sqrt{d}} \quad = \quad \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} f_{ij}\left(\ell_j^{\sqrt{d}}\right)$$

$$= \quad \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} \tilde{f}_{ij}(\ell_j)$$

$$= \quad \sum_{i}^{\mathrm{poly}(s,d)} \prod_{j=1}^{s} \prod_{k=1}^{d} \left(\ell_j - \zeta_{ijk}\right)$$

... a $\Sigma\Pi\Sigma$ circuit of $\mathrm{poly}(s,d)$ size and degree sd.

Putting it together

general circuit
of size s

Putting it together

general circuit
of size s $\longrightarrow$ $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ circuit
of size $s^{O(\sqrt{d})}$

Putting it together

general circuit of size s $\longrightarrow$ $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ circuit of size $s^{O(\sqrt{d})}$

$\downarrow$

$\Sigma\bigwedge^{\sqrt{d}}\Sigma\bigwedge^{\sqrt{d}}\Sigma$ circuits of size $s^{O(\sqrt{d})} \cdot 2^{O(\sqrt{d})}$

Putting it together

general circuit of size s $\longrightarrow$ $\Sigma\overset{\sqrt{d}}{\Pi}\Sigma\overset{\sqrt{d}}{\Pi}$ circuit of size $s^{O(\sqrt{d})}$

$\downarrow$

$\Sigma\overset{\sqrt{d}}{\bigwedge}\Sigma\overset{\sqrt{d}}{\bigwedge}\Sigma$ circuits of size $s^{O(\sqrt{d})}$

Putting it together

general circuit of size s

$\longrightarrow$ $\Sigma \prod^{\sqrt{d}} \Sigma \prod^{\sqrt{d}}$ circuit of size $s^{O(\sqrt{d})}$

$\downarrow$ $\Sigma \bigwedge^{\sqrt{d}} \Sigma \bigwedge^{\sqrt{d}} \Sigma$ circuits of size $s^{O(\sqrt{d})}$

$\longleftarrow$ $\Sigma\Pi\Sigma$ circuits of size $s^{O(\sqrt{d})}$

Putting it together

general circuit of size s $\longrightarrow$ $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ circuit of size $s^{O(\sqrt{d})}$

$\downarrow$

$\Sigma\Pi\Sigma$ circuits of size $s^{O(\sqrt{d})}$ $\longleftarrow$ $\Sigma\bigwedge^{\sqrt{d}}\Sigma\bigwedge^{\sqrt{d}}\Sigma$ circuits of size $s^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Putting it together

general hom. circuit of size s

$\longrightarrow$

$\Sigma\prod^{\sqrt{d}}\Sigma\prod^{\sqrt{d}}$ circuit of size $s^{O(\sqrt{d})}$

$\downarrow$

$\Sigma\bigwedge^{\sqrt{d}}\Sigma\bigwedge^{\sqrt{d}}\Sigma$ circuits of size $s^{O(\sqrt{d})}$

$\longrightarrow$

$\Sigma\Pi\Sigma$ circuits of size $s^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Putting it together

general hom. circuit of size s $\longrightarrow$ $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ hom. circuit of size $s^{O(\sqrt{d})}$

$\downarrow$

$\Sigma\bigwedge^{\sqrt{d}}\Sigma\bigwedge^{\sqrt{d}}\Sigma$ circuits of size $s^{O(\sqrt{d})}$ $\longrightarrow$ $\Sigma\Pi\Sigma$ circuits of size $s^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Putting it together

general hom. circuit of size s $\longrightarrow$ $\Sigma\prod^{\sqrt{d}}\Sigma\prod^{\sqrt{d}}$ hom. circuit of size $s^{O(\sqrt{d})}$

$\downarrow$

$\Sigma\bigwedge^{\sqrt{d}}\Sigma\bigwedge^{\sqrt{d}}\Sigma$ hom. circuits of size $s^{O(\sqrt{d})}$ $\longrightarrow$ $\Sigma\prod\Sigma$ circuits of size $s^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Putting it together

general hom. circuit of size s $\longrightarrow$ $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ hom. circuit of size $s^{O(\sqrt{d})}$

$\downarrow$

$\Sigma\bigwedge^{\sqrt{d}}\Sigma\bigwedge^{\sqrt{d}}\Sigma$ hom. circuits of size $s^{O(\sqrt{d})}$ $\longrightarrow$ $\Sigma\Pi\Sigma$ **non-hom.** circuits of size $s^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Putting it together

general hom. circuit of size s $\longrightarrow$ $\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$ hom. circuit of size $s^{O(\sqrt{d})}$

$\downarrow$

$\Sigma\bigwedge^{\sqrt{d}}\Sigma\bigwedge^{\sqrt{d}}\Sigma$ hom. circuits of size $s^{O(\sqrt{d})}$ $\longrightarrow$ $\Sigma\Pi\Sigma$ **non-hom.** circuits of size $s^{O(\sqrt{d})}$

Question: Where should one try to prove lower bounds?

Summary

- Depth reduction can manifest in many forms. Finding the right building block is sometimes crucial.
- A *slightly* different proof of [Tavenas] yields a possible useful building block for homogeneous formulas with more factors.
- *Maybe* we can get $n^{\Omega(\log n)}$ lower bounds via modified shifted-partials.
- Can we say something similar about $\Sigma\Pi\Sigma\Pi^{[t]}$ circuits obtained from ABPs?

Call for contributors

A git survey on arithmetic circuit lower bounds:

`https://github.com/dasarpmar/lowerbounds-survey/`

Dankeschön